Math 212 Spring 2006 **Exam #3 – Solution**

1. Let $f(x, y, z)$ be a scalar function, and let $\mathbf{F}(x, y, z)$ be a vector field. (Assume both f and $\mathbf{F}$ have continuous partial derivatives of all orders.) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$.

   (a) curl grad $f = \mathbf{0}$.

   Solution: **True**.

   (b) div grad $f = 0$.

   Solution: **False**. For example if $f(x, y, z) = x^2$, then grad $f = (2x, 0, 0)$ and div grad $f = 2$.

   (c) div curl $\mathbf{F} = 0$.

   Solution: **True**.

   (d) Let C be an oriented curve. The path integral of f along C does not change when the orientation of C is reversed.

   Solution: **True**.

   (e) Let C be an oriented curve. The line integral of $\mathbf{F}$ along C does not change when the orientation of C is reversed.

   Solution: **False**. The integral changes by a minus–sign.

   (f) The expression $\mathbf{u} \cdot \mathbf{v}$ is a vector.

   Solution: **False**.

   (g) The expression $\mathbf{u} \times \mathbf{v}$ is a vector.

   Solution: **True**.

   (h) The expression $(\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ is a vector.

   Solution: **True**. First note that $(\mathbf{v} \cdot \mathbf{w})$ is a scalar, so $(\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ is the scalar product of the vector $\mathbf{u}$ with the scalar $(\mathbf{v} \cdot \mathbf{w})$.

   (i) Let S be an oriented surface. The quantity $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is a vector.

   Solution: **False**.

   (j) Let S be an oriented surface. The quantity $\iint_S f\, dS$ is a vector.

   Solution: **False**.

2. Let $\mathbf{F}(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right)$.

   (a) Show curl $\mathbf{F} = (0, 0, 0)$.

   (b) Let C be the unit circle in the xy–plane, oriented **clockwise**. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$.

   (c) Using your answer from (b), explain why $\mathbf{F}$ is not a gradient field, even though curl $\mathbf{F} = (0, 0, 0)$.

Solution:

(a) This is a straight–forward computation:

$$\begin{aligned}
\text{curl } \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} \\
&= -\mathbf{i} \frac{\partial}{\partial z} \frac{x}{x^2+y^2} + \mathbf{j} \frac{\partial}{\partial z} \frac{-y}{x^2+y^2} + \mathbf{k} \left(\frac{\partial}{\partial x} \frac{x}{x^2+y^2} - \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} \right) \\
&= \mathbf{k} \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right) \\
&= (0, 0, 0).
\end{aligned}$$

(b) We first have to parametrize C. We take

$$\mathbf{c}(t) = (\cos(-t), \sin(-t)), t \in [0, 2\pi].$$

(note that we need the $-t$ to ensure that we go clockwise). We then compute

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_{t=0}^{t=2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\
&= \int_{t=0}^{t=2\pi} (-\sin(-t), \cos(-t), 0) \cdot (\sin(-t), -\cos(-t), 0) dt \\
&= \int_{t=0}^{t=2\pi} -1 dt = -2\pi.
\end{aligned}$$

(Note that one could also choose the usual clockwise parametrization $\mathbf{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$, and then changed the integral by a minus–sign.)

(c) If $\mathbf{F}$ was a gradient field, then any line integral over a closed curve would be zero. But in (b) we saw that this is not the case.

3. The surface S is parameterized by $\mathbf{\Phi}(u, v) = (e^u - 2, 2v + 3, 5 + u^2 + v^2)$ with $u, v \in \mathbb{R}$.

   (a) Determine the equation of the tangent plane to $(-1, 5, 6) \in S$.

   (b) Find all points on S for which the tangent plane is parallel to the xy–plane.

Solution:

(a) We compute

$$\begin{aligned} \mathbf{T}_u &= (e^u, 0, 2u) \\ \mathbf{T}_v &= (0, 2, 2v) \\ \mathbf{T}_u \times \mathbf{T}_v &= (-4u, -2ve^u, 2e^u). \end{aligned}$$

Now now note that $\Phi(u, v) = (-1, 5, 6)$ for $u = 0, v = 1$ (which can be seen by looking at the first two coordinates).

For $u = 0, v = 1$ we have

$$\mathbf{T}_u \times \mathbf{T}_v = (0, -2, 2).$$

This is the normal vector to the tangent plane at $(-1, 5, 6)$. The equation of the tangent plane is therefore given by

$$(x - (-1), y - 5, z - 6) \cdot (0, -2, 2) = 0$$

which simplifies to

$$-2y + 2z - 2 = 0.$$

(b) First note that the tangent plane is parallel to the xy–plane if the z–components of $\mathbf{T}_u$ and $\mathbf{T}_v$ are zero. But this means that $2u = 0$ and $2v = 0$. So the only possibility is $u = 0, v = 0$. The corresponding point on S is $\Phi(0, 0) = (-1, 3, 5)$.

4. Let $f(x, y) = \frac{1}{3}x^3 + y\sqrt{2} + 3$, and let D be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$. Let S be the surface given by the graph of $f(x, y)$ over D.

(a) Find a parametrization of S.

(b) Compute $\displaystyle\iint_S 4x^2\, dS$.

Solution:

(a) The parametrization is given by

$$\Phi(x, y) = (x, y, \frac{1}{3}x^3 + y\sqrt{2} + 3)$$

where the domain for x, y is given by D, i.e. by the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

(b) Let's first compute $||\mathbf{T}_x \times \mathbf{T}_y||$:

$$\begin{aligned} \mathbf{T}_x &= (1, 0, x^2) \\ \mathbf{T}_y &= (0, 1, \sqrt{2}) \\ \mathbf{T}_x \times \mathbf{T}_y &= (-x^2, -\sqrt{2}, 1) \\ ||\mathbf{T}_x \times \mathbf{T}_y|| &= \sqrt{3 + x^4}. \end{aligned}$$

We compute

$$\iint_S 4x^2\, dS = \int_{x=0}^{x=1} \int_{y=0}^{y=x} 4x^2 \sqrt{3 + x^4}\, dy dx = \int_{x=0}^{x=1} 4x^3 \sqrt{3 + x^4}\, dx.$$

Now let $u = 3 + x^4$ and we get

$$\int_{x=0}^{x=1} 4x^3\sqrt{3+x^4}\,dx = \int_{u=3}^{u=4} \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=3}^{u=4} = \frac{2}{3}(4^{\frac{3}{2}} - 3^{\frac{3}{2}}).$$

5. Consider the solid hemisphere formed by taking the portion of the unit ball with $y \geq 0$. Let S be the **surface** of this region (so that S is a hemisphere, together with a flat 'base' in the xz–plane). Find the flux of the vector field $\mathbf{V}(x, y, z) = -z\mathbf{i} + \mathbf{j} + x\mathbf{k}$ out of the surface S.

   You may find the following identity useful: $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$.

   Solution: We have to break this problem into two parts. We first integrate over the hemisphere, and then we integrate over the flat base.

   For the hemisphere we take the parametrization

$$\Phi(\theta,\phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$$

   with $\theta \in [0, \pi]$ and $\phi \in [0, \pi]$. We compute

$$\begin{aligned}
\mathbf{T}_\theta &= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\
\mathbf{T}_\phi &= (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi)) \\
\mathbf{T}_\theta \times \mathbf{T}_\phi &= (-\cos(\theta)\sin^2(\phi), -\sin(\theta)\sin^2(\phi), -\sin(\phi)\cos(\phi)) \\
&= -\sin(\phi)\,(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)).
\end{aligned}$$

   Note that $T_\theta \times T_\phi$ points inward (this can be seen by considering the last line and noticing that $-\sin(\phi)$ is negative). So it has the wrong orientation. We can fix this by putting a minus–sign into the formula. So the flux through the hemisphere is given by

$$\begin{aligned}
& -\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\pi} \mathbf{V}(\Phi(\theta,\phi)) \cdot (\mathbf{T}_\theta \times \mathbf{T}_\phi)\,d\phi d\theta \\
= & -\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\pi} (-\cos(\phi), 1, \cos(\theta)\sin(\phi)) \cdot (-\cos(\theta)\sin^2(\phi), -\sin(\theta)\sin^2(\phi), -\sin(\phi)\cos(\phi)\,d\phi d\theta \\
= & -\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\pi} -\sin(\theta)\sin^2(\phi)\,d\phi d\theta \\
= & -\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\pi} -\tfrac{1}{2}\sin(\theta)(1-\cos(2\phi))\,d\phi d\theta \\
= & -\int_{\theta=0}^{\theta=\pi} \left[-\tfrac{1}{2}\sin(\theta)(\phi - \tfrac{1}{2}\sin(2\phi))\right]_{\phi=0}^{\phi=\pi} d\theta \\
= & -\int_{\theta=0}^{\theta=\pi} -\tfrac{\pi}{2}\sin(\theta) d\theta \\
= & -\left[\tfrac{\pi}{2}\cos(\theta)\right]_{\theta=0}^{\theta=\pi} \\
= & \ \pi.
\end{aligned}$$

   Another way to approach this part of the problem is to see that $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the unit normal vector to the sphere at the point (x, y, z). Thus $\mathbf{V}\cdot\mathbf{n} = y$. To obtain the flux, one integrates this (as a scalar surface integral) in spherical coordinates. (It is a little bit faster this way.)

Now let's integrate over the base. The base lies in the xz–plane and the normal vector is $-\mathbf{j}$ (since outside is to the left), since the $\mathbf{j}$–component of $\mathbf{V}$ is constant one, we see that the flux integral is just -1 times the area of the base, which equals $-\pi$.

To compute the flux over S we have to add up the results from the hemisphere and the base, and we get that the flux equals $\pi - \pi = 0$.

A more formal approach is to find a parametrization again. We could take

$$\Phi(r,\theta) = (r\cos(\theta), 0, r\sin(\theta))$$

where $r \in [0,1]$ and $\theta \in [0, 2\pi]$. Then

$$\begin{aligned} \mathbf{T}_r &= (\cos(\theta), 0, \sin(\theta)) \\ \mathbf{T}_\theta &= (-r\sin(\theta), 0, r\cos(\theta)) \\ \mathbf{T}_r \times \mathbf{T}_\theta &= (0, -r, 0). \end{aligned}$$

This is the correct orientation, since $(0,-1,0)$ points outside. So we have:

$$\begin{aligned} \int\int \mathbf{V}\cdot d\mathbf{S} &= \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi} \mathbf{V}(\Phi(r,\theta))\cdot(\mathbf{T}_r\times\mathbf{T}_\theta)\, d\theta dr \\ &= \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi} (-r\sin(\theta), 1, r\cos(\theta))\cdot(0,-r,0)\, d\theta dr \\ &= \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi} -r\, d\theta dr = -\pi. \end{aligned}$$

6. Let $\mathbf{c}(t) = (1, -t^2, \cos t)$, $0 \le t \le \pi$. Evaluate

$$\int_{\mathbf{c}} \sin z\ dx - y^2\ dy + 3xz\ dz.$$

Solution: We compute

$$\begin{aligned} \int_{\mathbf{c}} \sin z\ dx - y^2\ dy + 3xz\ dz &= \int_{t=0}^{t=\pi} \sin(\cos(t))\tfrac{d}{dt}(1) - (-t^2)^2\tfrac{d}{dt}(-t^2) + 3\cos(t)\tfrac{d}{dt}(\cos(t))dt \\ &= \int_{t=0}^{t=\pi} 2t^5 - 3\cos(t)\sin(t))dt \\ &= [\tfrac{t^6}{3} + \tfrac{1}{2}3\cos(t)^2]_{t=0}^{t=\pi} \\ &= \tfrac{\pi^6}{3}. \end{aligned}$$